Pergamon

Int. J. Solids Structures Vol. 33, No. 29, pp. 4267–4291, 1996

0020–7683/96 $15.00 + .00

0020–7683(95)00266–9

EFFECTIVE ELASTOPLASTIC BEHAVIOR OF TWO-PHASE DUCTILE MATRIX COMPOSITES: A MICROMECHANICAL FRAMEWORK

J. W. JU and K. H. TSENG
Department of Civil and Environmental Engineering, University of California, Los Angeles, Los Angeles, CA 90095-1593, U.S.A.

(*Received* 14 *October* 1995)

Abstract—A micromechanics-based framework is presented to predict effective elastoplastic behavior of two-phase particle-reinforced ductile matrix composites (PRDMCs) containing many randomly dispersed elastic spherical inhomogeneities. Specifically, the inclusion phase (particle) is assumed to be elastic and the matrix phase is elastoplastic. A complete *second-order* formulation is presented based on the probabilistic spatial distribution of spherical particles, pairwise particle interactions and the ensemble-volume averaging procedure. Two non-equivalent formulations are considered in detail to derive the effective yield functions. In addition, the plastic flow rule and hardening law are postulated according to continuum plasticity and, together with the micromechanically derived effective yield function, are employed to characterize the plastic behavior of PRDMCs under three-dimensional arbitrary loading/unloading histories. Initial effective yield criteria for incompressible ductile matrix containing many randomly dispersed spherical voids are also studied. Furthermore, uniaxial elastoplastic stress–strain behavior of PRDMCs is investigated. Comparison between our theoretical uniaxial stress–strain predictions and experimental data for PRDMCs is also performed to illustrate the capability of the proposed framework. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

When ductile matrices are reinforced by elastic inclusions of high strength and high moduli, they lead to greater strength in shear and compression. Reinforcements could be continuous in the form of fibers or discontinuous in the form of particles or whiskers. Although continuous fiber-reinforced ductile matrix composites offer highly directional properties such as high specific stiffness along the reinforcement direction, particle-reinforced ductile matrix composites (PRDMCs) are widely used as they can exhibit nearly isotropic properties (if randomly oriented) and are often easier to process. See Ibrahim *et al.* (1991) for a general review of particle reinforced metal matrix composites.

The main objective of this paper is to predict effective elastoplastic behavior of two-phase particle reinforced ductile matrix composites based on mechanical properties of constituent phases, volume fractions, random spatial distributions and micro-geometries of particles. Furthermore, the particles are assumed to be *elastic* spheres (randomly dispersed in the matrix) and the ductile matrix behaves *elastoplastically* under arbitrary loading histories. All particles are assumed to be non-intersecting and embedded firmly in the matrix with perfect interfaces. Composites consisting of a metallic matrix reinforced by particles are examples of PRDMCs. The reinforcing particles, for example, could be carbides, nitrides, oxides, elemental materials, and so on. Existing studies on this subject have been primarily the "effective medium methods" such as the self-consistent method (e.g., Hutchinson (1970, 1976)) and the Mori–Tanaka method (e.g., Tandon and Weng (1988), Weng (1990), Li and Chen (1990), Lagoudas *et al.* (1991), and Bhattacharyya *et al.* (1993)). In addition, Suquet (1993) proposed mathematical bounds for metal matrix composites corresponding to some special cases such as power law or rigid-perfectly plastic materials. Most recently, Ju and Chen (1994a) proposed a micromechanical framework to predict the effective elastoplastic behavior of two-phase metal matrix random composites under arbitrary loading histories by considering the *first-order* (noninteracting) stress perturbations of elastic particles on the ductile matrix and the second-order relationship between the far-field stress σ^o and the ensemble-volume averaged stress $\bar{\sigma}$ (based on the work of Ju and Chen (1994b, c)).

In the self-consistent method, effects of particle interactions are approximated by embedding a single particle in an infinite "effective medium". Such an analysis, as pointed out by Tandon and Weng (1988), may lead to a significant overestimate of the overall yield strength of a PRDMC. On the other hand, within the Mori–Tanaka method, effects of particle interactions are taken into account by a "mean-field approximation". However, elastic properties of composites predicted by any effective medium methods are only dependent on geometries (i.e., shapes, orientations and volume fractions) of particles, and are *independent* of their spatial *locations* and distributions. Therefore, rigorously speaking, effective medium methods are more suitable for dilute or low concentrations of inclusions in which spatial locations and inter-particle interactions are not important.

According to the plasticity theory, the response at every *local* matrix point depends on its own spatial location and loading history. In order to obtain the *deterministic* overall behavior of a PRDMC, plastic field quantities (such as plastic strains and plastic hardening variables) must be recorded for every *local* point during the entire arbitrary loading history for any given particle configuration. Furthermore, hundreds of Monte Carlo simulations need to be performed to obtain the overall elastoplastic behavior of a *random* (not periodic) particle reinforced ductile matrix composite. This approach is precluded due to the complexity of random microstructures as well as the lack of exact microstructural information under normal situation. Therefore, statistical averaging methods have to be employed at the micromechanical level. Instead of exact deterministic solutions, a statistically representative (ensemble-volume averaged) effective elastoplastic formulation is pursued in the present study. The "local stress norm" needed in the matrix plasticity formulation is calculated analytically by a micromechanical approach which considers complete *second-order* pairwise inter-particle interactions. This is at variance with Ju and Chen's (1994a) formulation in which only the *first-order* (noninteracting) stress perturbations on the matrix points due to elastic particles are considered. Probabilistic ensemble average is subsequently applied in this paper to obtain a homogenized "plastic loading function". The plastic flow rule and hardening law are then postulated at the composite level based on continuum plasticity. Hence, complete *second-order* macroscopic effective elastoplastic constitutive models are established for PRDMCs.

For the special problem of an incompressible ductile matrix containing many randomly dispersed spherical *voids*, the effective yield criterion micromechanically derived in this paper is shown to predict a *finite* initial yield stress for the von Mises type (J_2) plasticity under purely deviatoric loadings. This demonstrates that the proposed approach is capable of, in an average sense, capturing the local stress perturbations due to the presence of and interactions among spherical voids. Moreover, when the interactions among voids are completely neglected, the effective initial yield stresses predicted by our method are identical to the upper bounds of Ponte Castaneda (1991) and the results of the energy method proposed by Qiu and Weng (1993).

In this study, the micromechanical approach of inter-particle interactions is combined with the continuum plasticity (von Mises model) to predict the effective elastoplastic behavior of a ductile matrix containing many randomly dispersed elastic spherical particles. Inter-particle interactions are considered for *both* the elastic and plastic sub-problems. A complete *second-order* formulation is proposed based on the pairwise particle interactions and the ensemble-volume averaging (homogenization) to construct the macroscopic (overall) yield functions for two-phase PRDMCs.

This paper is organized as follows. In Section 2, effective elastic moduli of two-phase composites containing randomly dispersed spherical particles are summarized based on Ju and Chen (1994b, c). In particular, relations between the stress/strain concentration factor tensors and effective elastic moduli are established. A second-order formulation is presented in Section 3 to account for particle interaction effects. Two non-equivalent formulations to derive the overall yield functions are considered in detail. In addition, the plastic flow rule and hardening law are postulated according to continuum plasticity to characterize the plastic behavior under *arbitrary* three-dimensional loading and unloading histories (in contrast to monotonic and proportional loadings assumed by most existing works in the micromechanics literature). Initial effective yield criteria for incompressible ductile matrix containing many identical spherical voids are presented in Section 4. The proposed non-interacting (first-order) and interacting (second-order) effective initial yield functions are compared with those proposed by Gurson (1977), Tvergaard (1981), Ponte Castaneda (1991), and Qiu and Weng (1993). Furthermore, uniaxial elastoplastic stress–strain behavior of PRDMCs is studied in Section 5. Our model predictions are compared with experimental data by Yang *et al.* (1991). In a forthcoming paper, three-dimensional computational return mapping algorithms, finite element implementations and effective elasto-viscoplastic (rate-dependent) behavior of PRDMCs will be presented.

2. EFFECTIVE ELASTIC MODULI OF TWO-PHASE COMPOSITES CONTAINING RANDOMLY DISPERSED SPHERICAL PARTICLES

Let us start by considering a composite consisting of an elastic matrix (phase 0) and randomly dispersed elastic spherical particles (phase 1) with distinct material properties. The two phases are perfectly bonded at interfaces. The relation between the stress σ and strain ε at any point x in the α-phase ($\alpha = 0$ or 1) are governed by

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}_\alpha : \boldsymbol{\varepsilon}(\mathbf{x})\,; \quad \boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{D}_\alpha : \boldsymbol{\sigma}(\mathbf{x}) \tag{1}$$

where : denotes the tensor contraction, and $\mathbf{C}_\alpha$ and $\mathbf{D}_\alpha$ are the elastic stiffness and compliance tensors, respectively.

2.1. *Relations between concentration factor tensors and macroscopic moduli*

By taking the volume average (denoted by an overbar) of eqn (1) over the sub-domain occupied by the α-phase, we obtained

$$\bar{\boldsymbol{\sigma}}_\alpha = \mathbf{C}_\alpha : \bar{\boldsymbol{\varepsilon}}_\alpha\,; \quad \bar{\boldsymbol{\varepsilon}}_\alpha = \mathbf{D}_\alpha : \bar{\boldsymbol{\sigma}}_\alpha \tag{2}$$

At the macroscopic level, overall elastic stiffness $\mathbf{C}_*$ and compliance $\mathbf{D}_*$ moduli are defined as the relations between global averages of stress and strain:

$$\bar{\boldsymbol{\sigma}} = \mathbf{C}_* : \bar{\boldsymbol{\varepsilon}}; \quad \bar{\boldsymbol{\varepsilon}} = \mathbf{D}_* : \bar{\boldsymbol{\sigma}} \tag{3}$$

Ideally, if the stress and strain fields of a composite can be solved deterministically, the local stress and strain for the α-phase would be related to the global averages as follows

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{B}_\alpha(\mathbf{x}) : \bar{\boldsymbol{\sigma}}\,; \quad \boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{A}_\alpha(\mathbf{x}) : \bar{\boldsymbol{\varepsilon}} \tag{4}$$

in which $\mathbf{A}_\alpha(\mathbf{x})$ and $\mathbf{B}_\alpha(\mathbf{x})$ are the *local* strain and stress concentration factor tensors, respectively, for the α-phase. It is apparent that exact solutions would only be possible if detailed microstructural information, such as the locations and configurations of particles, and exact solutions of many-particle interaction problems are available. In practical situations, it is often not possible to obtain this microstructural information. Furthermore, exact solutions of many-particle interaction problems are intractable. However, averaging eqn (4) over the α-phase provides a tractable avenue based on the volume-averaged quantities rather than the actual local solutions.

Following Hill (1963) and Dvorak (1991), we write

$$\bar{\boldsymbol{\sigma}}_\alpha = \mathbf{B}_\alpha : \bar{\boldsymbol{\sigma}}\,; \quad \bar{\boldsymbol{\varepsilon}}_\alpha = \mathbf{A}_\alpha : \bar{\boldsymbol{\varepsilon}} \tag{5}$$

where $\mathbf{A}_\alpha$ and $\mathbf{B}_\alpha$ are the volume-averaged *strain* and *stress* concentration factor tensors, respectively, for the α-phase. The relations between concentration factors for the two phases are obtained by using eqn (5) and the volume-averaging procedure:

$$\phi_0 \mathbf{B}_0 + \phi_1 \mathbf{B}_1 = \mathbf{I}; \quad \phi_0 \mathbf{A}_0 + \phi_1 \mathbf{A}_1 = \mathbf{I} \tag{6}$$

where $\mathbf{I}$ is the fourth rank identity tensor and ϕ_α denotes the α-phase volume fraction.

Moreover, substituting eqn (2) into (6) and using (3) yields

$$\bar{\boldsymbol{\sigma}} = \phi_0 \mathbf{C}_0 : \bar{\boldsymbol{\varepsilon}}_0 + \phi_1 \mathbf{C}_1 : \bar{\boldsymbol{\varepsilon}}_1 ; \quad \bar{\boldsymbol{\varepsilon}} = \phi_0 \mathbf{D}_0 : \bar{\boldsymbol{\sigma}}_0 + \phi_1 \mathbf{D}_1 : \bar{\boldsymbol{\sigma}}_1 \tag{7}$$

and hence

$$\mathbf{C}_* = \phi_0 \mathbf{C}_0 \cdot \mathbf{A}_0 + \phi_1 \mathbf{C}_1 \cdot \mathbf{A}_1 ; \quad \mathbf{D}_* = \phi_0 \mathbf{D}_0 \cdot \mathbf{B}_0 + \phi_1 \mathbf{D}_1 \cdot \mathbf{B}_1 \tag{8}$$

Therefore, macroscopic elastic moduli are expressed in terms of volume fractions, phase moduli, and concentration factor tensors of both phases. Alternatively, employing eqn (6), one can write ($\alpha \neq \beta$):

$$\mathbf{C}_* = \mathbf{C}_\alpha + \phi_\beta (\mathbf{C}_\beta - \mathbf{C}_\alpha) \cdot \mathbf{A}_\beta ; \quad \mathbf{D}_* = \mathbf{D}_\alpha + \phi_\beta (\mathbf{D}_\beta - \mathbf{D}_\alpha) \cdot \mathbf{B}_\beta \tag{9}$$

Simple manipulations then lead to the following relations:

$$\mathbf{B}_\alpha \cdot \mathbf{C}_* = \mathbf{C}_\alpha \cdot \mathbf{A}_\alpha ; \quad \mathbf{A}_\alpha \cdot \mathbf{D}_* = \mathbf{D}_\alpha \cdot \mathbf{B}_\alpha \tag{10}$$

2.2. *Inter-particle interactions and ensemble-volume averaged fields*

In Ju and Chen (1994c), it was shown that the approximate ensemble-volume averaged eigenstrain $\langle \bar{\boldsymbol{\varepsilon}}^* \rangle$ (accounting for *pairwise* spherical particle interaction) is related to the noninteracting eigenstrain solution $\boldsymbol{\varepsilon}^{*o}$ as follows

$$\langle \bar{\boldsymbol{\varepsilon}}^* \rangle = \boldsymbol{\Gamma} : \boldsymbol{\varepsilon}^{*o} \tag{11}$$

where the components of the *isotropic* tensor Γ are defined as

$$\Gamma_{ijkl} = \gamma_1 \delta_{ij} \delta_{kl} + \gamma_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \tag{12}$$

in which (assuming the uniform "radical distribution function")

$$\gamma_1 = \frac{5\phi}{4\beta^2} \left\{ -2(1-\nu_0) - 5\nu_0^2 - \frac{4\alpha}{3\alpha+2\beta}(1+\nu_0)(1-2\nu_0) \right\} \tag{13}$$

$$\gamma_2 = \tfrac{1}{2} + \frac{5\phi}{8\beta^2} \left\{ 11(1-\nu_0) + 5\nu_0^2 - \frac{3\alpha}{3\alpha+2\beta}(1+\nu_0)(1-2\nu_0) \right\} \tag{14}$$

and

$$\alpha = 2(5\nu_0 - 1) + 10(1-\nu_0)\left(\frac{\kappa_0}{\kappa_1 - \kappa_0} - \frac{\mu_0}{\mu_1 - \mu_0} \right) \tag{15}$$

$$\beta = 2(4 - 5\nu_0) + 15(1-\nu_0)\frac{\mu_0}{\mu_1 - \mu_0} \tag{16}$$

Following Ju and Chen (1994b), it can be shown that the averaged strain $\bar{\boldsymbol{\varepsilon}}$, the uniform remote strain $\boldsymbol{\varepsilon}^o$ and the averaged eigenstrain $\bar{\boldsymbol{\varepsilon}}^*$ are related by (dropping the ensemble notation):

$$\bar{\varepsilon} = \varepsilon^o + \phi \mathbf{s} : \bar{\varepsilon}^* \tag{17}$$

where subscript 1 is omitted in $\bar{\varepsilon}^*$ and the components of the Eshelby tensor $\mathbf{s}$ (for a spherical inclusion embedded in an isotropic linear elastic and infinite matrix) are

$$s_{ijkl} = \frac{1}{15(1-v_0)}\{(5v_0-1)\delta_{ij}\delta_{kl}+(4-5v_0)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\} \tag{18}$$

Therefore, we arrive at

$$\bar{\varepsilon}^* = \mathbf{B} : \bar{\varepsilon} \equiv [\Gamma \cdot (-\mathbf{A} - \mathbf{s} + \phi \mathbf{s} \cdot \Gamma)^{-1}] : \bar{\varepsilon} \tag{19}$$

where the fourth-rank tensor $\mathbf{A}$ is defined as

$$\mathbf{A} \equiv [\mathbf{C}_1 - \mathbf{C}_0]^{-1} \cdot \mathbf{C}_0 \tag{20}$$

Employing the Eshelby's equivalence principle, averaging quantities over the particle domain V_1 and recalling eqn (19), we obtain the strain concentration factor tensor $\mathbf{A}_1$:

$$\bar{\varepsilon}_1 = -\mathbf{A} : \bar{\varepsilon}^* = -[\mathbf{A} \cdot \mathbf{B}] : \bar{\varepsilon} \equiv \mathbf{A}_1 : \bar{\varepsilon} \tag{21}$$

2.3. *Effective elastic moduli of two-phase composites*

Effective elastic moduli of two-phase composites containing randomly distributed *identical* elastic spheres are readily available by substituting eqn (21) into (9):

$$\mathbf{C}_* = \mathbf{C}_0 \cdot \{\mathbf{I} - \phi \Gamma \cdot (-\mathbf{A} - \mathbf{s} + \phi \mathbf{s} \cdot \Gamma)^{-1}\} \tag{22}$$

which recovers eqn (53) in Ju and Chen (1994c). Effective bulk modulus κ_* and shear modulus μ_* can be explicitly evaluated as

$$\kappa_* = \kappa_0 \left\{1 + \frac{30(1-v_0)\phi(3\gamma_1+2\gamma_2)}{3\alpha+2\beta-10(1+v_0)\phi(3\gamma_1+2\gamma_2)}\right\} \tag{23}$$

$$\mu_* = \mu_0 \left\{1 + \frac{30(1-v_0)\phi\gamma_2}{\beta-4(4-5v_0)\phi\gamma_2}\right\} \tag{24}$$

Equations (23) and (24) are valid for any arbitrary two-point isotropic "radial distribution function". Moreover, the effective Young's modulus E_* and Poisson's ratio v_* of a particulate composite are obtained through the following relations

$$E_* = \frac{9\kappa_*\mu_*}{3\kappa_*+\mu_*} \tag{25}$$

$$v_* = \frac{3\kappa_*-2\mu_*}{6\kappa_*+2\mu_*} \tag{26}$$

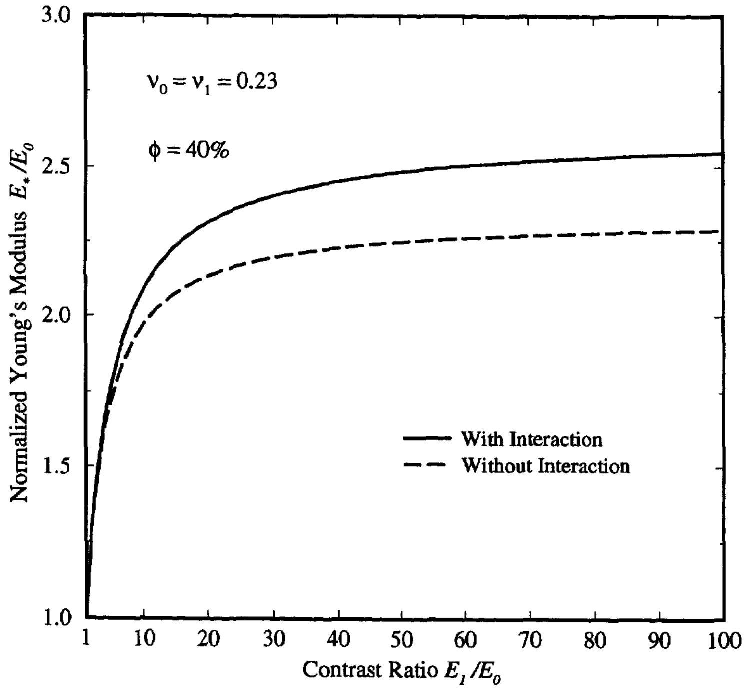

Fig. 1. The normalized effective Young's modulus E_*/E_0 vs the elastic contrast ratio E_1/E_0 for elastic composites containing dispersed harder spheres.

To illustrate effects of elastic inter-particle interaction, Figs 1 and 2 (harder and softer inclusions, respectively) display the normalized effective Young's moduli E_*/E_0 vs the "contrast ratios" E_1/E_0. The particle volume fraction ϕ is taken as 0.4 and the Poisson's ratio is assumed to be 0.23 for both constituent phases. It is evident from Fig. 1 (or Fig. 2) that particle interactions significantly affect overall elastic moduli when the contrast ratios are high (or low).

3. EFFECTIVE ELASTOPLASTIC BEHAVIOR OF PRDMCs

3.1. *Overview*

Let us consider a two-phase composite consisting of elastic spheres (with bulk and shear moduli κ_1 and μ_1, respectively) dispersed in an elastoplastic matrix (with elastic bulk and shear moduli κ_0 and μ_0, respectively). For simplicity, the von Mises yield criterion with an isotropic hardening law is assumed here. Extension of the present framework to general yield criterion and general hardening law, however, is straightforward. Accordingly, at any matrix material point, the stress $\boldsymbol{\sigma}$ and the equivalent plastic strain $\bar{e}^p$ must satisfy the following yield function:

$$F(\boldsymbol{\sigma}, \bar{e}^p) = H(\boldsymbol{\sigma}) - K^2(\bar{e}^p) \leqslant 0 \tag{27}$$

in which $K(\bar{e}^p)$ is the isotropic hardening function of the matrix-only material. Furthermore, $H(\boldsymbol{\sigma}) \equiv \boldsymbol{\sigma} : \mathbf{I}_d : \boldsymbol{\sigma}$ signifies the square of the deviatoric stress norm, where $\mathbf{I}_d$ denotes the *deviatoric* part of the fourth rank identity tensor $\mathbf{I}$, i.e.,

$$\mathbf{I}_d \equiv \mathbf{I} - \tfrac{1}{3}\mathbf{1} \otimes \mathbf{1} \tag{28}$$

in which $\mathbf{1}$ represents the second rank identity tensor.

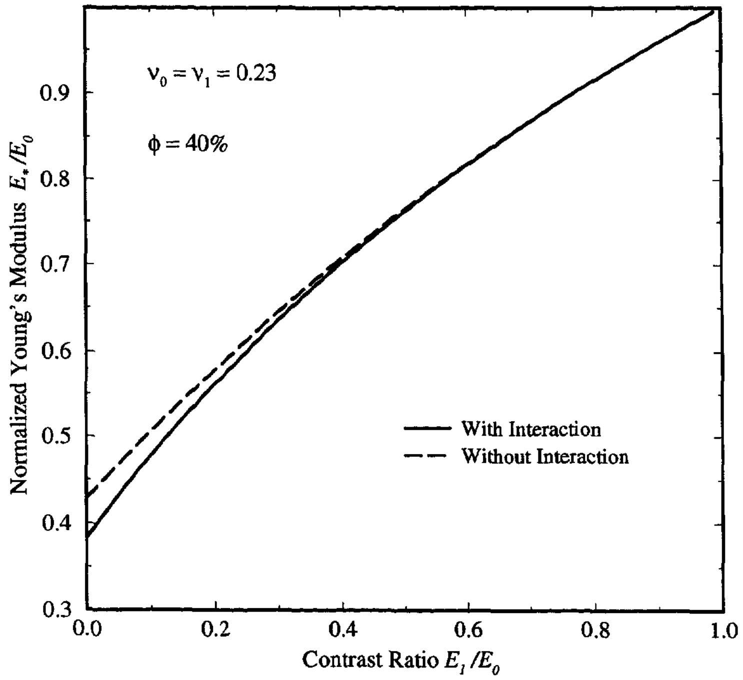

Fig. 2. The normalized effective Young's modulus E_*/E_0 vs the elastic contrast ratio E_1/E_0 for elastic composites containing dispersed softer spheres.

According to the theory of continuum plasticity, the total strain ε can be decomposed into two parts:

$$\varepsilon = \varepsilon^e + \varepsilon^p \tag{29}$$

where ε^e denotes the elastic strain of the matrix and particles and ε^p represents the stress-free plastic strain in the plastic matrix only. In order to solve the elastoplastic response exactly, the stress at any local point has to be solved and then used to determine the plastic response through the local yield criterion for all possible configurations. This approach is in general infeasible due to the complexity of statistical and microstructural information. Therefore, a framework in which an *ensemble averaged* yield criterion is constructed for the entire composite is proposed. The methodology is generally parallel to the work of Ju and Chen (1994a) in which only the first order effects are considered in the formulation of effective plastic response. It is noted that, in Ju and Chen (1994a), the interactions among particles are neglected in the process of collecting the perturbations of stresses at a local matrix point for the purpose of predicting the plastic behavior although effective elastic properties with pairwise inter-particle interactions are utilized. By contrast, a technique which approximately accounts for the pairwise interaction among particles while collecting the local stress perturbations in the plastic matrix is proposed in this section. As a result, the present work renders a complete second order elastoplastic formulation for PRDMCs, which incorporates inter-particle interactions in both the elastic and plastic responses.

3.2. *A second-order formulation accounting for particle interaction effects*

For simplicity, small strains are assumed and therefore the statistical microstructure of particles embedded in a ductile matrix remains essentially the same. Hence, the microstructure is assumed to be statistically homogeneous and isotropic with a virtually constant particle volume fraction during the deformation process. Furthermore, in what follows, particles are considered as elastic spheres of uniform size.

Following Ju and Chen (1994a), we denote by $H(\mathbf{x} \mid \mathscr{G})$ the square of the "current stress norm" at the local point $\mathbf{x}$, which determines the plastic strain in a PRDMC for a given particle configuration $\mathscr{G}$. Since there is no plastic strain in the elastic particles, $H(\mathbf{x} \mid \mathscr{G})$ can be written as

$$H(\mathbf{x}|\mathscr{G}) = \begin{cases} \sigma(\mathbf{x}|\mathscr{G}) : \mathbf{I}_d : \boldsymbol{\sigma}(\mathbf{x}|\mathscr{G}), & \text{if } \mathbf{x} \text{ is in the matrix;} \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

In addition, $\langle H \rangle_m(\mathbf{x})$ is defined as the ensemble average of $H(\mathbf{x} \mid \mathscr{G})$ over all possible realizations where $\mathbf{x}$ is in the matrix phase. Let $P(\mathscr{G})$ be the probability density function for finding the particle configuration $\mathscr{G}$ in the composite, $\langle H \rangle_m(\mathbf{x})$ can be obtained by integrating H over all possible particle configurations (for a point $\mathbf{x}$ in the matrix).

$$\langle H \rangle_m(\mathbf{x}) = H^o + \int_{\mathscr{G}} \{H(\mathbf{x}|\mathscr{G}) - H^o\} P(\mathscr{G}) \, \mathrm{d}\mathscr{G} \tag{31}$$

where H^o is the square of the far-field stress norm in the matrix:

$$H^o = \boldsymbol{\sigma}^o : \mathbf{I}_d : \boldsymbol{\sigma}^o \tag{32}$$

Moreover, the total stress at any point $\mathbf{x}$ in the matrix is the superposition of the far-field stress $\boldsymbol{\sigma}^o$ and the perturbed stress $\boldsymbol{\sigma}'$ due to the presence of the particles:

$$\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}^o + \boldsymbol{\sigma}'(\mathbf{x}) \tag{33}$$

in which $\boldsymbol{\sigma}^o$ and $\boldsymbol{\sigma}'$ are defined as

$$\boldsymbol{\sigma}^o \equiv \mathbf{C}_0 : \boldsymbol{\varepsilon}^o \tag{34}$$

$$\boldsymbol{\sigma}'(\mathbf{x}) \equiv \mathbf{C}_0 : \int_V \mathbf{G}(\mathbf{x}-\mathbf{x}') : \boldsymbol{\varepsilon}^*(\mathbf{x}') \, \mathrm{d}\mathbf{x}' \tag{35}$$

where $\boldsymbol{\varepsilon}^o$ is the elastic strain field induced by the far-field loading, $\boldsymbol{\varepsilon}^*$ denotes the *elastic* eigenstrain in the particle phase, $\mathbf{C}_0$ denotes the fourth rank elasticity tensor of the matrix, and V is the statistically representative volume element (infinitely large compared with inhomogeneities and *without* any prescribed displacement boundary conditions along infinite exterior boundaries). It is noted that eqn (35) represents the method of Green's function. In indicial notation, the components of the fourth rank tensor $\mathbf{G}$ read

$$G_{ijkl}(\mathbf{x}-\mathbf{x}') = \frac{1}{8\pi(1-\nu_0)r^3} F_{ijkl}(-15, 3\nu_0, 3, 3-6\nu_0, -1+2\nu_0, 1-2\nu_0) \tag{36}$$

where $\mathbf{r} \equiv \mathbf{x}-\mathbf{x}'$, $r \equiv \|\mathbf{r}\|$, and ν_0 is the Poisson's ratio of the matrix material. The components of the fourth-rank tensor $\mathbf{F}$—which depends on six scalar quantities B_1, B_2, B_3, B_4, B_5, B_6—are defined by:

$$\begin{aligned} F_{ijkl}(B_m) \equiv\ & B_1 n_i n_j n_k n_l + B_2(\delta_{ik} n_j n_l + \delta_{il} n_j n_k + \delta_{jk} n_i n_l + \delta_{jl} n_i n_k) \\ & + B_3 \delta_{ij} n_k n_l + B_4 \delta_{kl} n_i n_j + B_5 \delta_{ij} \delta_{kl} + B_6(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \end{aligned} \tag{37}$$

with the unit normal vector $\mathbf{n} \equiv \mathbf{r}/r$ and index $m = 1$–6. It is observed that the above expression of Green's function is fundamentally *different* from the "modified Green's function" considered by Mazilu (1972) and Kroner (1990) in which the *prescribed* displacements or Green's function values on the (finite or infinite) exterior boundaries are *homogeneous* (zero).

The unknown elastic eigenstrain $\boldsymbol{\varepsilon}^*(\mathbf{x})$ within the particles can be solved by the integral equation obtained on the basis of the celebrated Eshelby's equivalence principle (Eshelby (1957)), which guarantees that the equilibrium conditions in both the matrix and particle phases and the boundary conditions at the particle–matrix interfaces are satisfied exactly. The result is

$$-\mathbf{A}:\boldsymbol{\varepsilon}^*(\mathbf{x}) = \boldsymbol{\varepsilon}^o + \int_V \mathbf{G}(\mathbf{x}-\mathbf{x}'):\boldsymbol{\varepsilon}^*(\mathbf{x}')\,\mathrm{d}\mathbf{x}' \tag{38}$$

where the fourth-rank tensor $\mathbf{A}$ is defined in eqn (20).

The first-order approximation approach proposed in Ju and Chen (1994a) is based upon the work of Eshelby (1957); i.e., the (elastic) eigenstrain for a single inclusion is *uniform* for the interior points of an isolated (*noninteracting*) inclusion. Consequently, the constant (elastic) eigenstrain can be moved out of the integration in eqn (35). Accordingly, the perturbed stress for any matrix point $\mathbf{x}$ due to an isolated elastic spherical particle centered at $\mathbf{x}_1$ becomes

$$\boldsymbol{\sigma}'(\mathbf{x}|\mathbf{x}_1) = [\mathbf{C}_0\cdot\bar{\mathbf{G}}(\mathbf{x}-\mathbf{x}_1)]:\boldsymbol{\varepsilon}^{*o} \tag{39}$$

where

$$\bar{\mathbf{G}}(\mathbf{x}-\mathbf{x}_1) \equiv \int_{\Omega_1} \mathbf{G}(\mathbf{x}-\mathbf{x}')\,\mathrm{d}\mathbf{x}' \tag{40}$$

for $\mathbf{x}\notin\Omega_1$ in which Ω_1 is the particle domain centered at $\mathbf{x}_1$; or by the definition of eqn (37):

$$\bar{\mathbf{G}}(\mathbf{r}) = \frac{1}{30(1-\nu_0)}(\rho^3\mathbf{H}^1+\rho^5\mathbf{H}^2) \tag{41}$$

The components of $\mathbf{H}^1$ and $\mathbf{H}^2$ are given by

$$\mathbf{H}^1_{ijkl}(\mathbf{r}) \equiv 5\mathbf{F}_{ijkl}(-15, 3\nu_0\, 3, 3-6\nu_0, -1+2\nu_0\, 1-2\nu_0) \tag{42}$$

$$\mathbf{H}^2_{ijkl}(\mathbf{r}) \equiv 3\mathbf{F}_{ijkl}(35, -5, -5, -5, 1, 1) \tag{43}$$

where $\mathbf{r} = \mathbf{x}-\mathbf{x}_1$, $\rho = a/r$, and a = the radius of a particle. Furthermore, the elastic noninteracting eigenstrain $\boldsymbol{\varepsilon}^{*o}$ (corresponding to the *single* inclusion problem) in eqn (39) is given by (see, e.g., Ju and Chen (1994b, 1994c))

$$\boldsymbol{\varepsilon}^{*o} = -(\mathbf{A}+\mathbf{s})^{-1}:\boldsymbol{\varepsilon}^o \tag{44}$$

where $\mathbf{s}$ is Eshelby's tensor for a spherical inclusion. In general, Eshelby's tensor depends on the Poisson's ratio of the matrix and the shape of the ellipsoidal inclusion; see Mura (1987) for details. Explicitly, the components of $\mathbf{s}$ for a spherical particle take the form

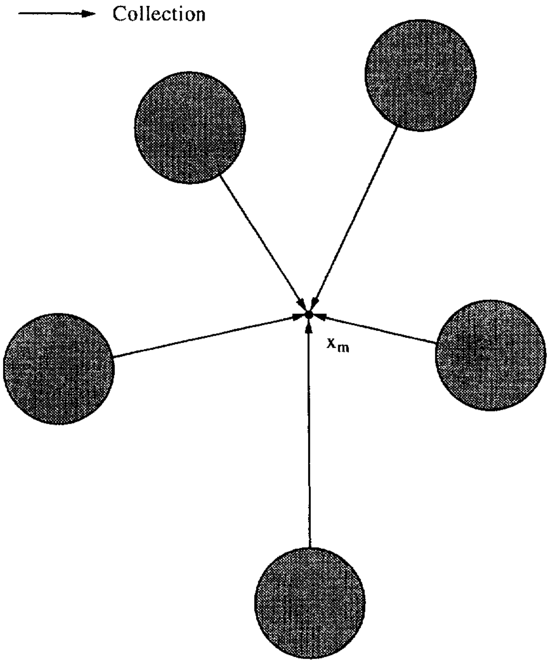

Fig. 3. The local matrix point $\mathbf{x}_m$ collects stress perturbations due to surrounding particles without near-field inter-particle interaction.

$$s_{ijkl} = \frac{1}{15(1-\nu_0)}\{(5\nu_0-1)\delta_{ij}\delta_{kl}+(4-5\nu_0)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\} \tag{45}$$

This process is illustrated in Fig. 3 which shows that at a matrix point $\mathbf{x}_m$, the surrounding particles are treated as *isolated* (noninteracting) sources of perturbation. A matrix point simply *collects* the perturbation from all *noninteracting* particles one by one. In the absence of exact solution for many particle interaction problems, this first-order approximation provides a simple way to account for the perturbations on a matrix point from the particles. However, it is noted that the effective *elastic* moduli utilized in Ju and Chen (1994a) is based on the second-order formulation where the inter-particle interactions are accounted for through an approximation which collects the interactions between any pairs of particles.

For a complete second-order formulation, any two-particle pair should interact *first* and then the matrix point collects the perturbation based on the results of pairwise interaction. Unfortunately, the domain for centers of the interacting particle pair is complex. Mathematically, for a matrix point $\mathbf{x}_m$, the possible locations for the centers of a pair of particles can be expressed as

$$\{(\mathbf{x}_1,\mathbf{x}_2)|\,|\mathbf{x}_m-\mathbf{x}_1|>a,\quad |\mathbf{x}_m-\mathbf{x}_2|>a,\quad |\mathbf{x}_1-\mathbf{x}_2|>2a\} \tag{46}$$

which ensures that particles do not penetrate each other and the matrix point is not occupied by any particle. This domain makes the analytical solution for the ensemble-average process intractable. Consequently, *approximate* solutions which render analytical results are in order.

In this paper, an approach motivated by the solution of the ensemble-volume averaged eigenstrain for the inter-particle interaction problem is proposed; c.f. eqn (11) in Section 2.2. Since the particles under consideration are dispersed uniformly in the matrix, it is very reasonable to infer that eqn (11) is the average eigenstrain for any particle in the composite. Therefore, this eigenstrain can be used to calculate the ensemble-volume averaged perturbation to the matrix point (due to the existence and interaction of particles). As depicted

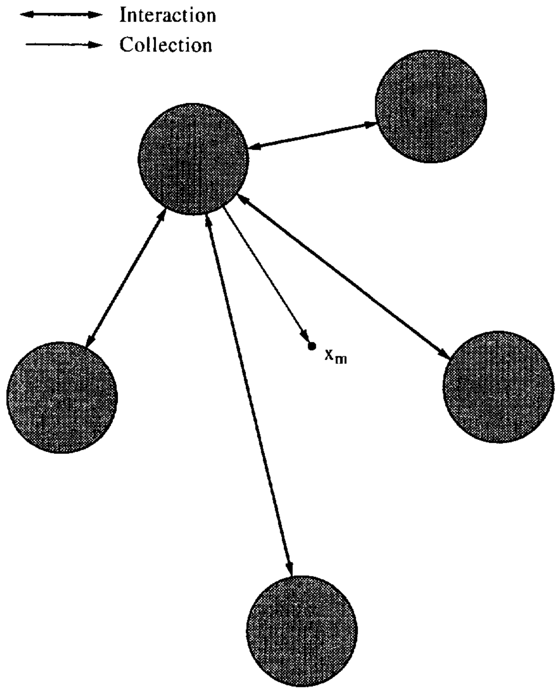

Fig. 4. The local matrix point $\mathbf{x}_m$ collects stress perturbations due to surrounding particles with pairwise inter-particle interaction.

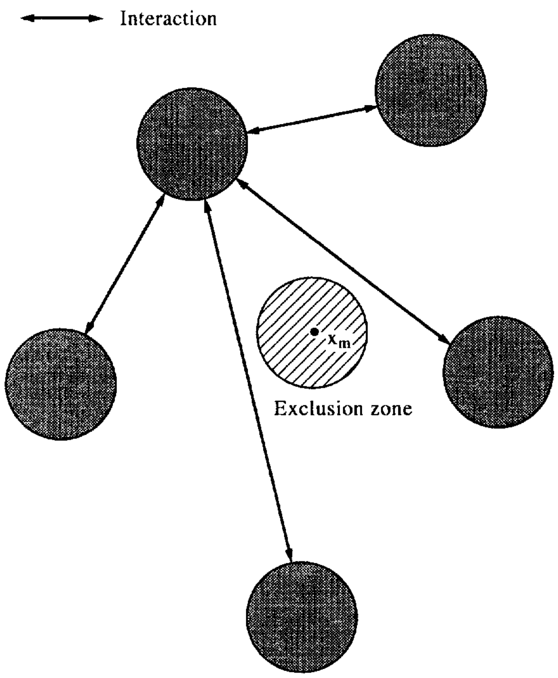

Fig. 5. The exclusion zone for a local matrix point $\mathbf{x}_m$, in which the center of a random particle should not be located.

in Fig. 4, conceptually, any given particle interacts pairwise with all surrounding particles in the proposed complete second-order formulation. The matrix point then collects the perturbations from all interacting particles. Rigorously speaking, there exists a very small (of the radius a) exclusion zone which excludes the possibility of having the center of any particle located within the zone; see Fig. 5 for illustration. Since the exclusion zone is so small and insignificant in comparison with the entire (infinitely large) statistical averaging domain, it will be neglected in our proposed treatment here. As a result, the proposed approximate treatment renders an analytical and compact formulation which is attractive.

Furthermore, by using the ensemble-volume averaged eigenstrain given in eqn (11), the stress perturbation in eqn (35) can be rephrased as

$$\boldsymbol{\sigma}'(\mathbf{x}|\mathbf{x}_1) = [\mathbf{C}_0 \cdot \bar{\mathbf{G}}(\mathbf{x}-\mathbf{x}_1) \cdot \boldsymbol{\Gamma}] : \boldsymbol{\varepsilon}^{*o} \tag{47}$$

within the framework of the proposed second-order pairwise particle interaction during the perturbation collection process at any matrix point. It is emphasized that the new second-order approximation given in eqn (47) is fundamentally different from the previous first-order approximation (Ju and Chen (1994a)) given in eqn (39). Clearly, if one totally neglects the particle interaction effects, then $\boldsymbol{\Gamma}$ in (47) would reduce to $\mathbf{I}$ and therefore (39) would be recovered.

3.3. *A second-order formulation of effective elastoplastic behavior of two-phase PRDMCs*

Since a matrix point receives the perturbations from particles *after* the interaction effects are accounted for, the ensemble-average stress norm for any matrix point $\mathbf{x}$ can be evaluated by collecting the current stress norm perturbed due to a particle centered at $\mathbf{x}_1$ and averaging over all possible locations of $\mathbf{x}_1$. Mathematically, we write

$$\langle H \rangle_m(\mathbf{x}) \cong H^o + \int_{|\mathbf{x}-\mathbf{x}_1|>a} \{H(\mathbf{x}|\mathbf{x}_1) - H^o\} P(\mathbf{x}_1)\, \mathrm{d}\mathbf{x}_1 + \cdots \tag{48}$$

where $P(\mathbf{x}_1)$ denotes the probability density function for finding a particle centered at $\mathbf{x}_1$. In this paper, $P(\mathbf{x}_1)$ is assumed to be statistically homogeneous, isotropic and uniform, and takes the form $P(\mathbf{x}_1) = N/V$, where N is the total number of particles dispersed in a volume V. Moreover, due to the assumption of statistical isotropy and uniformity, eqn (48) can be recast into a more convenient form:

$$\langle H \rangle_m(\mathbf{x}) \cong H^o + \frac{N}{V} \int_{r>a} \mathrm{d}r \int_{A(r)} \{H(\mathbf{r}) - H^o\}\, \mathrm{d}A + \cdots \tag{49}$$

where $A(r)$ is a spherical surface of radius r.

With the help of the two identities eqns (28)–(29) in Ju and Chen (1994a) and the perturbed stress given in eqn (47), we arrive at the ensemble-averaged current stress norm at any matrix point:

$$\langle H \rangle_m(\mathbf{x}) = \boldsymbol{\sigma}^o : \mathbf{T} : \boldsymbol{\sigma}^o \tag{50}$$

Here the components of the positive definite fourth-rank tensor $\mathbf{T}$ are given by

$$T_{ijkl} = T_1 \delta_{ij} \delta_{kl} + T_2 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \tag{51}$$

with

$$3T_1 + 2T_2 = 200(1-2\nu_0)^2 \frac{(3\gamma_1 + 2\gamma_2)^2}{(3\alpha + 2\beta)^2} \phi \tag{52}$$

$$T_2 = \tfrac{1}{2} + (23 - 50\nu_0 + 35\nu_0^2) \frac{4\gamma_2^2}{\beta^2} \phi \tag{53}$$

in which the particle volume fraction ϕ is defined as $\phi \equiv 4\pi a^3/3\, N/V$. It should be noted that the newly proposed second-order (interacting) expressions for T_1 and T_2 in eqns (52) and (53) are very different from the first-order (noninteracting) expressions previously given in eqns (32)–(33) in Ju and Chen (1994a).

It is interesting to observe that if one allows the volume fraction ϕ to go to zero in eqns (52) and (53), the tensor $\mathbf{T}$ reduces to the fourth-rank deviatoric identity tensor $\mathbf{I}_d$. Consequently, the local stress norm defined in eqn (50) reduces to the second deviatoric stress invariant J_2 which is employed to define the yield function for the classical von Mises yield criterion.

The stress norm given in eqn (50) is in terms of the far-field stress $\boldsymbol{\sigma}^o$. Alternatively, the ensemble-averaged current stress norm at a matrix point can be expressed in terms of the macroscopic stress $\bar{\boldsymbol{\sigma}}$. Following Ju and Chen (1994a), the relation between the far-field stress $\boldsymbol{\sigma}^o$ and the macroscopic stress $\bar{\boldsymbol{\sigma}}$ is given by

$$\boldsymbol{\sigma}^o = \mathbf{P}:\bar{\boldsymbol{\sigma}} \tag{54}$$

where the components of $\mathbf{P}$ read

$$P_{ijkl} = P_1\delta_{ij}\delta_{kl} + P_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{55}$$

with

$$3P_1 + 2P_2 = \frac{1}{1+a\phi} \tag{56}$$

$$P_2 = \frac{1}{2(1+b\phi)} \tag{57}$$

and the coefficients a and b are given by:

$$a = 20(1-2\nu_0)\frac{3\gamma_1+2\gamma_2}{3\alpha+2\beta} \tag{58}$$

$$b = (7-5\nu_0)\frac{2\gamma_2}{\beta} \tag{59}$$

It is noteworthy to mention that $\mathbf{P}$ in eqn (54) includes the elastic pairwise particle interaction effects and is employed in both the present formulation and the previous formulation (Ju and Chen (1994a)). Nevertheless, the present complete second-order formulation also considers the pairwise particle interaction effects in the collecting process of matrix stress perturbations through eqns (47) and (52)–(53), whereas Ju and Chen (1994a) totally neglect particle interaction effects in the perturbation collection process as evidenced by eqn (39) herein and eqns (32)–(33) therein.

Combination of eqns (54) and (50) then leads to the alternative expression for the ensemble-averaged current stress norm in a matrix point:

$$\langle H \rangle_m(\mathbf{x}) = \bar{\boldsymbol{\sigma}}:\bar{\mathbf{T}}:\bar{\boldsymbol{\sigma}} \tag{60}$$

where the positive definite fourth-rank tensor $\bar{\mathbf{T}}$ is defined as

$$\bar{\mathbf{T}} \equiv \mathbf{P}\cdot\mathbf{T}\cdot\mathbf{P} \tag{61}$$

By carrying out the lengthy algebra, the components of $\bar{\mathbf{T}}$ are explicitly given by

$$\bar{T}_{ijkl} = \bar{T}_1\delta_{ij}\delta_{kl} + \bar{T}_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{62}$$

where

$$3\bar{T}_1 + 2\bar{T}_2 = \frac{3T_1 + 2T_2}{(1 + a\phi)^2} \tag{63}$$

$$\bar{T}_2 = \frac{T_2}{(1 + b\phi)^2} \tag{64}$$

It is observed again that the ensemble-averaged current stress norm in a matrix point given in eqn (60) reduces to the classical J_2 stress invariant upon the substitution of $\phi = 0$ into eqns (63) and (64). In fact, the tensor $\bar{\mathbf{T}}$ reduces to the fourth-rank deviatoric identity tensor $\mathbf{I}_d$ for $\phi = 0$.

In what follows, we will present two alternative (and non-equivalent) formulations to represent the ensemble-volume averaged yield function for a two-phase ductile matrix composite.

(*a*) *Formulation I*: *matrix average approach.* The first formulation which we will consider here is based on the concept that plastic yielding and plastic flow occur only in the *matrix* material. Therefore, one can regard the two-phase composite as "plastic" overall when the ensemble-volume averaged "current stress norm" in the matrix reaches a certain level. From eqn (60), it is observed that the ensemble-averaged stress norm is uniform for any point in the matrix. Accordingly, the effective (ensemble-volume averaged) yield criterion can be proposed as

$$\bar{F}_m = \bar{\boldsymbol{\sigma}} : \bar{\mathbf{T}} : \bar{\boldsymbol{\sigma}} - K_m^2(\bar{e}_m^p) \tag{65}$$

where $\bar{e}_m^p$ is the ensemble-volume averaged equivalent plastic strain of the matrix and $K_m(\bar{e}_m^p)$ denotes the isotropic hardening function for the matrix material. It should be noted that the effective yield function is pressure dependent and not of the von Mises type any more. Moreover, for simplicity, we assume that the overall flow rule for the matrix is associative. Therefore, the averaged plastic strain rate of the matrix can be postulated as

$$\dot{\bar{\boldsymbol{\varepsilon}}}_m^p = \dot{\lambda}\frac{\partial \bar{F}_m}{\partial \bar{\boldsymbol{\sigma}}} = 2\dot{\lambda}\bar{\mathbf{T}} : \bar{\boldsymbol{\sigma}} \tag{66}$$

Here, $\dot{\lambda}$ denotes the plastic consistency parameter.

In addition, inspired by the structure of the micromechanically derived stress norm, the averaged equivalent plastic strain rate for the matrix is defined as

$$\dot{\bar{e}}_m^p \equiv \sqrt{\tfrac{2}{3}\dot{\bar{\boldsymbol{\varepsilon}}}_m^p : \bar{\mathbf{T}}^{-1} : \dot{\bar{\boldsymbol{\varepsilon}}}_m^p} = 2\dot{\lambda}\sqrt{\tfrac{2}{3}\bar{\boldsymbol{\sigma}} : \bar{\mathbf{T}} : \bar{\boldsymbol{\sigma}}} \tag{67}$$

The $\dot{\lambda}$ together with the yield function $\bar{F}_m$ must obey the Kuhn–Tucker conditions

$$\dot{\lambda} \geqslant 0, \quad \bar{F}_m \leqslant 0, \quad \dot{\lambda}\bar{F}_m = 0, \quad \dot{\lambda}\dot{\bar{F}}_m = 0 \tag{68}$$

The Kuhn–Tucker conditions define the state of loading and unloading.

It is noted that the ensemble-volume averaged yield function in eqn (65), the averaged plastic flow rule in eqn (66), the equivalent plastic strain rate in eqn (67), and the Kuhn–Tucker conditions in eqn (68) completely characterize the effective plasticity formulation for the matrix material with any isotropic hardening function $K_m(\bar{e}_m^p)$. Straightforward extension of the proposed model can be made to accommodate the *kinematic hardening*. For simplicity, the following power-law type isotropic hardening function is utilized as an example in the subsequent study:

$$K_m(\bar{e}^p_m) = \sqrt{\tfrac{2}{3}}\{\sigma_y + h(\bar{e}^p_m)^q\} \tag{69}$$

where σ_y denotes the initial yield stress, and h and q are the linear and the exponential isotropic hardening parameters, respectively, of the matrix material.

In addition, the overall effective plastic strain rate $\dot{\bar{\boldsymbol{\varepsilon}}}^p$ for the two-phase composite as a whole can be related to the effective plastic strain rate $\dot{\bar{\boldsymbol{\varepsilon}}}^p_m$ for the matrix material as follows

$$\dot{\bar{\boldsymbol{\varepsilon}}}^p = (1-\phi)\mathbf{B}_0 : \dot{\bar{\boldsymbol{\varepsilon}}}^p_m \tag{70}$$

Here, the volume-averaged stress concentration factor tensor $\mathbf{B}_0$ takes the form (cf. Section 2)

$$\mathbf{B}_0 = \mathbf{C}_0 \cdot \mathbf{A}_0 \cdot \mathbf{C}_*^{-1} \tag{71}$$

in which

$$\mathbf{A}_0 = \frac{1}{1-\phi_1}\mathbf{I} - \frac{\phi_1}{1-\phi_1}\mathbf{A}_1 \tag{72}$$

and $\mathbf{A}_1 = -\mathbf{A}\cdot\mathbf{B}$ (as defined in eqn (21)).

(*b*) *Formulation II*: *overall two-phase average approach*. Alternatively, the ensemble-averaged "current stress norm" for any point $\mathbf{x}$ in a two-phase particulate composite can be defined as:

$$\sqrt{\langle H\rangle(\mathbf{x})} = (1-\phi)\sqrt{\bar{\boldsymbol{\sigma}}:\bar{\mathbf{T}}:\bar{\boldsymbol{\sigma}}} \tag{73}$$

Consequently, the overall effective yield function for the two-phase PRDMC can be proposed as

$$\bar{F} = (1-\phi)^2\bar{\boldsymbol{\sigma}}:\bar{\mathbf{T}}:\bar{\boldsymbol{\sigma}} - K^2(\bar{e}^p) \tag{74}$$

with the isotropic hardening function $K(\bar{e}^p)$ for the two-phase composite (different from $K_m(\bar{e}^p_m)$ in "Formulation I"). Again, we note that the effective yield function is *pressure dependent* and not of the von Mises type. Moreover, the effective ensemble-volume averaged plastic strain rate for the PRDMC can be expressed as

$$\dot{\bar{\boldsymbol{\varepsilon}}}^p = \dot{\lambda}\frac{\partial \bar{F}}{\partial \bar{\boldsymbol{\sigma}}} = 2(1-\phi)^2\dot{\lambda}\bar{\mathbf{T}}:\bar{\boldsymbol{\sigma}} \tag{75}$$

Similar to "Formulation I", the effective equivalent plastic strain rate for the composite is defined as

$$\dot{\bar{e}}^p \equiv \sqrt{\tfrac{2}{3}\dot{\bar{\boldsymbol{\varepsilon}}}^p:\bar{\mathbf{T}}^{-1}:\dot{\bar{\boldsymbol{\varepsilon}}}^p} = 2(1-\phi)^2\dot{\lambda}\sqrt{\tfrac{2}{3}\bar{\boldsymbol{\sigma}}:\bar{\mathbf{T}}:\bar{\boldsymbol{\sigma}}} \tag{76}$$

The Kuhn–Tucker conditions can be expressed similar to "Formulation I". Furthermore, the isotropic hardening function reads

$$K(\bar{e}^p) = \sqrt{\tfrac{2}{3}}\{\sigma_y + h(\bar{e}^p)^q\} \tag{77}$$

where σ_y denotes the initial yield stress, and h and q signify the linear and exponential isotropic hardening parameters, respectively, for the two-phase composite. It is emphasized that these parameters are at variance with those in "Formulation I" in which only the matrix average is considered.

4. INITIAL YIELD CRITERIA FOR INCOMPRESSIBLE DUCTILE MATRIX CONTAINING MANY RANDOMLY DISPERSED IDENTICAL SPHERICAL VOIDS

To illustrate the capability of the proposed effective elastoplastic framework, we will consider a special problem in this section—the prediction of *initial yield* stresses for an elastically incompressible and perfectly plastic J_2-type ductile matrix containing many randomly dispersed identical spherical voids at various volume fractions. This special problem has attracted interest from many researchers. For example, Gurson (1977) presented a study based on dilute and periodic array of noninteracting voids; Tvergaard (1981, 1982) modified Gurson's work (1977) to account for void interactions based on finite element results (not micromechanics); Ponte Castaneda (1991) presented mathematical upper bounds for porous ductile metals; and Qiu and Weng (1993) derived a yield criterion for ductile metals containing spheroidal inclusions through an energy approach.

In particular, the simplified problem under consideration is a metal material which is elastically incompressible and perfectly plastic with the J_2-flow (within the plastic domain), and contains many randomly dispersed identical spherical voids. Clearly, there is nothing inside the voids and hence the bulk and shear moduli are zero for voids. Therefore, it is reasonable to employ "Formulation I" and assume that the porous metal yields as the ensemble-volume averaged stress norm of the matrix phase reaches a certain level. As a consequence, the following yield criterion is employed (see eqn (65)):

$$\bar{F} = \bar{\boldsymbol{\sigma}} : \bar{\mathbf{T}} : \bar{\boldsymbol{\sigma}} - \tfrac{2}{3}\sigma_y^2 \tag{78}$$

in which the averaged matrix yield radius is taken as $K_m = \sqrt{(2/3)}\sigma_y$.

Letting both the bulk and shear moduli of voids, κ_1 and μ_1, vanish and the Poisson's ratio of the matrix, v_0, equal 1/2, we obtain $\alpha = 3$, $\beta = -4.5$, $1-2v_0/3\alpha+2\beta = -1/20$ and the following expressions for this special problem

$$3T_1 + 2T_2 = \frac{\phi}{2}(3\gamma_1 + 2\gamma_2)^2 \tag{79}$$

$$T_2 = \tfrac{1}{2} + \frac{4\gamma_2^2}{3}\phi \tag{80}$$

and

$$3\gamma_1 + 2\gamma_2 = 1 + \tfrac{5}{24}\phi \tag{81}$$

$$\gamma_2 = \tfrac{1}{2} + \tfrac{11}{48}\phi \tag{82}$$

with pairwise interaction effects accounted for.

Furthermore, the total averaged stress can be split into two parts:

$$\bar{\sigma}_{ij} = \bar{s}_{ij} + \bar{\sigma}\delta_{ij} \tag{83}$$

in which the hydrostatic stress $\bar{\sigma}$ and deviatoric stress $\bar{s}_{ij}$ are defined as

$$\bar{\sigma} \equiv \tfrac{1}{3}\bar{\sigma}_{kk}, \quad \text{and} \quad \bar{s}_{ij} \equiv \bar{\sigma}_{ij} - \tfrac{1}{3}\bar{\sigma}_{kk}\delta_{ij} \tag{84}$$

With these definitions, the initial yield criterion given in eqn (78) can be recast into the following form:

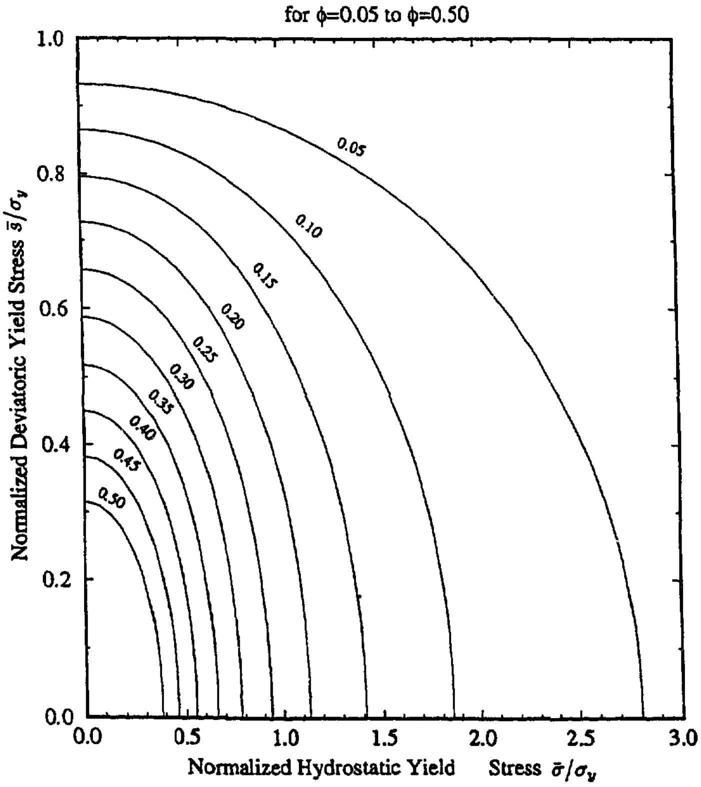

Fig. 6. The normalized yield surfaces for void volume fractions ϕ varying from 0.05 to 0.5.

$$\bar{F} = 9(3\bar{T}_1 + 2\bar{T}_2)\left(\frac{\bar{\sigma}}{\sigma_y}\right)^2 + 4\bar{T}_2\left(\frac{\bar{s}}{\sigma_y}\right)^2 - 2 = 0 \tag{85}$$

where the definition of the deviatoric stress norm reads

$$\bar{s} \equiv \sqrt{\tfrac{3}{2}\bar{s}_{ij}\bar{s}_{ij}} \tag{86}$$

It is noted that eqn (85) represents an initial yield surface at a specified void volume fraction ϕ in the $\bar{\sigma}-\bar{s}$ space. The initial yield surfaces for void volume concentrations ϕ varying from 0.05 to 0.5 are plotted in Fig. 6 which shows that the porous metal yields at a lower level as the volume fraction of voids ϕ increases.

As discussed earlier, the yield function corresponding to the complete first-order noninteracting formulation (inter-void interaction not considered at all) can be easily obtained by the following operations

$$\gamma_1 \to 0, \quad \text{and} \quad \gamma_2 \to \tfrac{1}{2} \tag{87}$$

which transforms eqn (85) into

$$\bar{F} = \frac{9\phi}{4(1+\frac{2}{3}\phi)}\left(\frac{\bar{\sigma}}{\sigma_y}\right)^2 + \left(\frac{\bar{s}}{\sigma_y}\right)^2 - \frac{(1-\phi)^2}{1+\frac{2}{3}\phi} = 0 \tag{88}$$

It is interesting to note that eqn (88) is *identical* to the initial yield function given in eqn (27) in Qiu and Weng (1993) and Ponte Castaneda's (1991) upper bound. That is, our

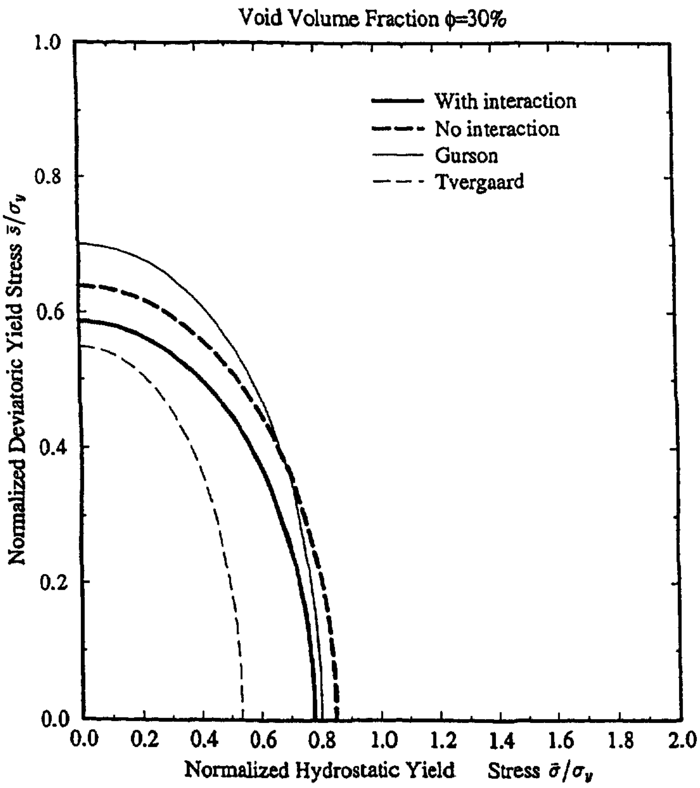

Fig. 7. Comparison of normalized yield functions predicted by four different models for the void volume fraction $\phi = 30\%$.

completely *noninteracting* solution in this case is actually the same as the upper bound solution of Ponte Castaneda (1991) and the energy approach of Qiu and Weng (1993).

In order to further compare the initial yield stresses predicted by the proposed framework and other methods, initial yield functions denoted by $\bar{F}_G$ and $\bar{F}_T$ for the Gurson's (1977) and Tvergaard's (1981) models, respectively, are taken from Qiu and Weng (1993):

$$\bar{F}_G = 2\phi \cosh\left(\frac{3\bar{\sigma}}{2\sigma_y}\right) + \left(\frac{\bar{s}}{\sigma_y}\right)^2 - (1+\phi^2) = 0 \tag{89}$$

$$\bar{F}_T = 2q_1\phi \cosh\left(\frac{3q_2\bar{\sigma}}{2\sigma_y}\right) + \left(\frac{\bar{s}}{\sigma_y}\right)^2 - (1+q_3\phi^2) = 0 \tag{90}$$

where $q_1 = 1.5$, $q_2 = 1$, and $q_3 = q_1^2 = 2.25$ are suggested by Tvergaard (1981). Initial yield functions predicted by different methods for void volume fractions of 30% and 40% are depicted in Figs 7 and 8, respectively. It is observed that void interactions result in *lower* initial yield stresses and the difference between the interacting (complete second-order) and the noninteracting (complete first-order) predictions increase as the void volume fraction increases. Figures 7 and 8 also demonstrate that interactions among voids bring our predicted initial yield stresses closer to those of Tvergaard's (1981) modification.

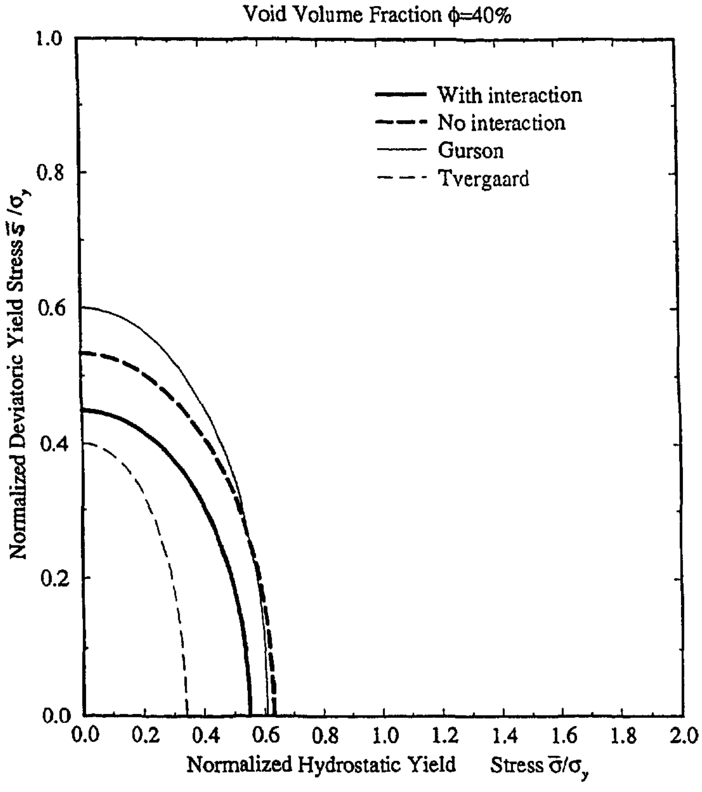

Fig. 8. Comparison of normalized yield functions predicted by four different models for the void volume fraction $\phi = 40\%$.

In addition, if the applied loading is purely hydrostatic, the equations characterizing the initial yield stresses for the four models under consideration are

$$\text{With interaction:}\quad \frac{\bar{\sigma}}{\sigma_y} = \frac{2(1-\phi-\frac{5}{24}\phi^2)}{3\sqrt{\phi}(1+\frac{5}{24}\phi)} \tag{91}$$

$$\text{No interaction:}\quad \frac{\bar{\sigma}}{\sigma_y} = \frac{2(1-\phi)}{3\sqrt{\phi}} \tag{92}$$

$$\text{Gurson:}\quad \frac{\bar{\sigma}}{\sigma_y} = \tfrac{2}{3}\ln\frac{1}{\phi} \tag{93}$$

$$\text{Tvergaard:}\quad \frac{\bar{\sigma}}{\sigma_y} = \tfrac{2}{3}\ln\frac{2}{3\phi} \tag{94}$$

The curves corresponding to the initial yield stresses vs void volume fractions for the four models are rendered in Fig. 9. Since the von Mises yield criterion is assumed for the matrix material, all four curves approach infinity as the void volume fraction approaches zero. When the void volume fraction is non-zero, the presence and interactions due to voids perturb the stress field and hence the local stresses are no longer hydrostatic. The proposed micromechanics-based models are capable of capturing these features.

Similarly, for purely *deviatoric* loading, the initial yield stresses are plotted in Fig. 10 based on the following four different predictions:

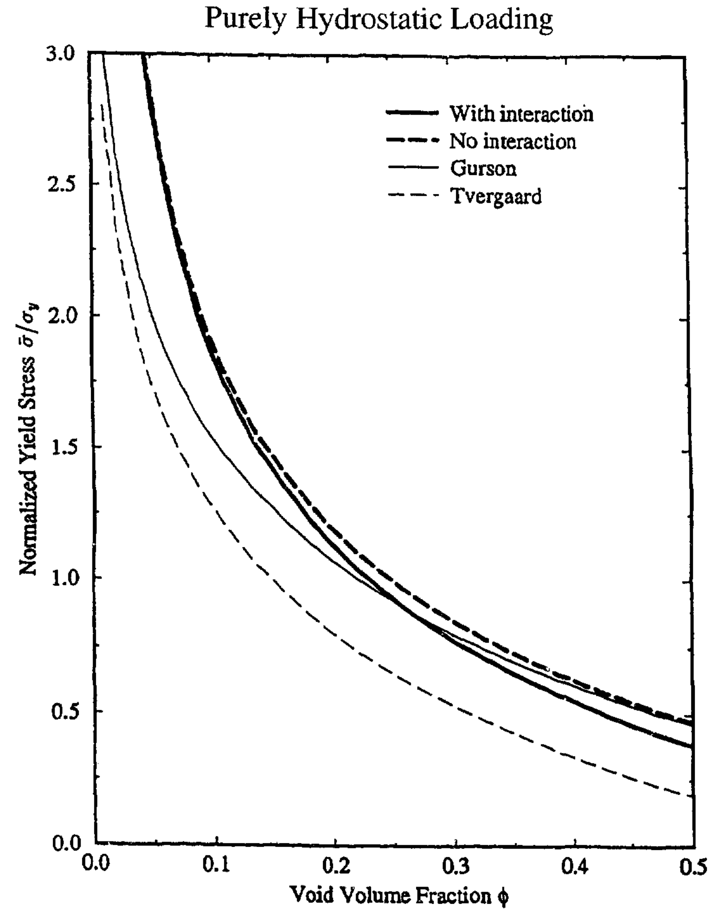

Fig. 9. Comparison of normalized yield stresses vs void volume fractions predicted by four different models under purely hydrostatic loading.

$$\text{With interaction:} \quad \frac{\bar{s}}{\sigma_y} = \frac{1-\phi+\frac{11}{24}\phi^2}{\sqrt{1+\frac{2}{3}\phi(1+\frac{11}{24}\phi)^2}} \tag{95}$$

$$\text{No interaction:} \quad \frac{\bar{s}}{\sigma_y} = \frac{1-\phi}{\sqrt{1+\frac{2}{3}\phi}} \tag{96}$$

$$\text{Gurson:} \quad \frac{\bar{s}}{\sigma_y} = 1-\phi \tag{97}$$

$$\text{Tvergaard:} \quad \frac{\bar{s}}{\sigma_y} = 1-\frac{3}{2}\phi \tag{98}$$

As can be seen from Fig. 10, the two curves corresponding to the proposed interacting and non-interacting models are bounded between the curves corresponding to Gurson's and Tvergaard's models. Again, the difference between the proposed interacting and non-interacting models increases as ϕ increases.

5. UNIAXIAL ELASTOPLASTIC STRESS-STRAIN RELATIONSHIP FOR PRDMCs

From experimental evidence, it is observed that plastic properties (initial yield stresses and plastic hardening constants) of a matrix material may *change* as the particle (or void)

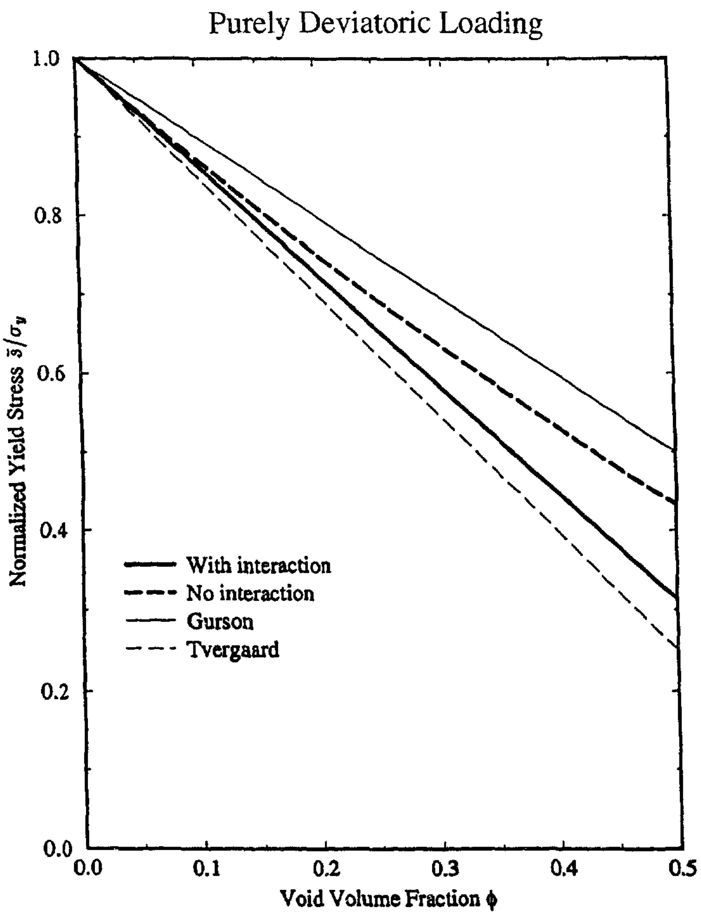

Fig. 10. Comparison of normalized yield stresses vs void volume fractions predicted by four different models under purely deviatoric loading.

volume fraction ϕ increases. That is, the existence and interactions of particles *affect* the plastic properties of the *matrix* material. The proposed "Formulation I" (effective *matrix* yield criterion) and the methods proposed by Gurson (1977), Ponte Castaneda (1991) and Qiu and Weng (1993) cannot capture this feature. On the other hand, the proposed "Formulation II" (effective *overall* yield criterion) can characterize the changes in plastic properties of the matrix and the composite due to the additional factor $(1-\phi)$ in eqn (74).

In order to illustrate the proposed micromechanics-based elastoplastic constitutive model for PRDMCs, let us consider the example of *uniaxial* stress loadings in which the applied macroscopic stress $\bar{\boldsymbol{\sigma}}$ can be written as

$$\bar{\sigma}_{11} = 0, \quad \text{all other } \bar{\sigma}_{ij} = 0. \tag{99}$$

With the simple isotropic hardening law described by eqn (77), the overall yield function reads

$$\bar{F}(\bar{\boldsymbol{\sigma}}, \bar{e}^p) = (1-\phi)^2 \bar{\boldsymbol{\sigma}} : \bar{\mathbf{T}} : \bar{\boldsymbol{\sigma}} - \tfrac{2}{3}\{\sigma_y + h(\bar{e}^p)^q\}^2 \tag{100}$$

Substituting eqn (99) into eqn (100), the effective yield function for the special case of uniaxial loading is obtained as

$$\bar{F}(\bar{\sigma}_{11}, \bar{e}^p) = (1-\phi)^2(\bar{T}_1+2\bar{T}_2)\bar{\sigma}_{11}^2 - \tfrac{2}{3}\{\sigma_y + h(\bar{e}^p)^q\}^2 \tag{101}$$

The macroscopic incremental plastic strain rate defined by eqn (75) becomes

$$\Delta\bar{\varepsilon}^p = 2(1-\phi)^2\,\Delta\lambda\bar{\sigma}_{11}\begin{pmatrix} \bar{T}_1+2\bar{T}_2 & 0 & 0 \\ 0 & \bar{T}_1 & 0 \\ 0 & 0 & \bar{T}_1 \end{pmatrix} \tag{102}$$

for any stress beyond the initial yielding. Similarly, the incremental equivalent plastic strain can be written as

$$\Delta\bar{e}^p = 2(1-\phi)^2\,\Delta\lambda|\bar{\sigma}_{11}|\sqrt{\tfrac{2}{3}(\bar{T}_1+2\bar{T}_2)} \tag{103}$$

From the linear theory of elasticity, the macroscopic incremental elastic strain takes the form

$$\Delta\bar{\varepsilon}^e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\nu_* & 0 \\ 0 & 0 & -\nu_* \end{pmatrix}\frac{\Delta\bar{\sigma}_{11}}{E_*} \tag{104}$$

Furthermore, as given in eqn (29), the total incremental strain is the sum of the elastic incremental strain and plastic incremental strain.

In the case of a *monotonic* uniaxial loading, the overall uniaxial stress–strain relation can be obtained by integrating eqns (102) and (104) as follows:

$$\bar{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\nu_* & 0 \\ 0 & 0 & -\nu_* \end{pmatrix}\frac{\Delta\bar{\sigma}_{11}}{E_*} + 2(1-\phi)^2\lambda\bar{\sigma}_{11}\begin{pmatrix} \bar{T}_1+2\bar{T}_2 & 0 & 0 \\ 0 & \bar{T}_1 & 0 \\ 0 & 0 & \bar{T}_1 \end{pmatrix} \tag{105}$$

where the positive parameter $\lambda = \Sigma\,\Delta\lambda$ is solved from the nonlinear equation obtained by enforcing the plastic consistency condition $\bar{F} = 0$. Since only the uniaxial loading is under consideration, the nonlinear equation reads (cf. eqn (103))

$$(1-\phi)^2(\bar{T}_1+2\bar{T}_2)\bar{\sigma}_{11}^2 = \tfrac{2}{3}\{\sigma_y + h[2(1-\phi)^2\lambda\sqrt{\tfrac{2}{3}(\bar{T}_2+2\bar{T}_2)}|\bar{\sigma}_{11}|]^q\}^2 \tag{106}$$

To demonstrate the capability of the proposed framework, the predictions from our "Formulation II" with particle interaction effects are compared with the experimental data reported by Yang *et al.* (1991). In their experiments, uniaxial stress–strain curves were recorded for the Al/4Mg alloy reinforced with SiC particles. Large reinforcements tend to crack, especially upon tensile loading, leading to a degradation in the strength of composites (Yang *et al.*, 1991). Hence, experimental results of composites obtained in compression containing small particles provide better data for our model evaluation. Furthermore, the aspect ratios of particulates reported in experimental data are less than 2 : 1 and are not too important for randomly oriented particulates. The elastic moduli $E_0 = 75$ GPa and $\nu_0 = 0.33$ are used for the matrix phase and the elastic constants for the SiC particles are $E_1 = 420$ GPa and $\nu_1 = 0.17$. It is noted that the analytical model employed by Yang *et al.* (1991) to describe the measured results is not the same as the isotropic hardening law we use here. Therefore, the plastic parameters σ_y, h and q (of the matrix) for our effective plastic hardening law were not reported in Yang *et al.* (1991).

In order to estimate the optimal values of plastic parameters for the simple hardening law employed, data points are sampled from the experimental stress–strain curves and used to obtain an optimal set of plastic parameters (σ_y, h, q) by *minimizing* the sum of the squares

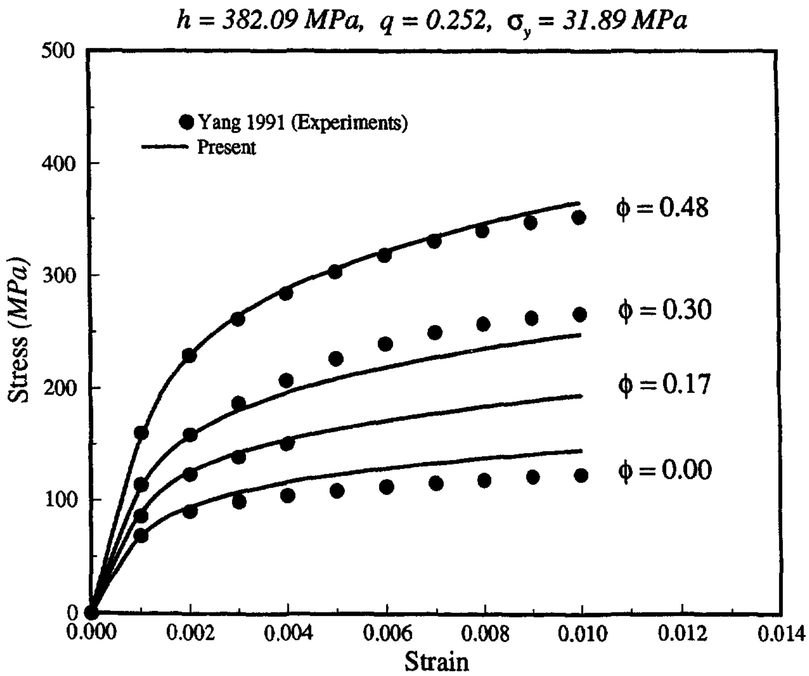

Fig. 11. Overall uniaxial stress–strain relation of PRDMCs for various $\phi = 0$, 0.17, 0.3 and 0.48. The solid lines correspond to the present predictions and solid circles correspond to experimental data of Yang *et al.* (1991).

of differences between the predicted and measured stresses at all data points. Let us assume that N data points are sampled and σ_i denotes the *experimental* value of stress corresponding to the strain ε_i at the sampling point. For each ε_i, the stress predicted by our model with the yet undetermined parameters σ_y, h and q is denoted by σ_i^*. The objective of the optimization is to minimize the sum of the squares of differences between the stress predictions and measurements. Symbolically, the least-square optimization function can be written as (with σ_y, h and q positive)

$$\min \sum_{i=1}^{N} (\sigma_i^* - \sigma_i)^2 \tag{107}$$

Apparently, this is a nonlinear least-square constrained minimization problem. A number of iterative numerical algorithms for searching the minimum of the foregoing objective function are discussed in Luenberger (1984). The well known modified Levenberg–Marquardt method is employed to search for the minimum in this study. See Levenberg (1944) and Marquardt (1963) for more details, as well as Ju *et al.* (1987) and Simo *et al.* (1988) for a summary of the algorithm. This method is basically a combination of the inverse-Hessian method and the deepest descent method.

The optimal values of the plastic parameters (for the simple hardening law) are thus obtained through the aforementioned minimization algorithm based on the data points shown in Fig. 11 which displays four uniaxial stress–strain curves for $\phi = 0$, 0.17, 0.3 and 0.48. The values of σ_y, h and q are fitted as $\sigma_y = 31.89$ MPa, $h = 382.09$ MPa, and $q = 0.252$. On the basis of these plastic constants, our "Formulation II" is exercised to render model predictions for the four uniaxial tests, as shown in Fig. 11. From these comparisons, it is seen that the proposed model performs very well for the matrix-only material and three different volume fractions of particles.

6. CONCLUSION

A micromechanics-based framework is presented in this paper to predict effective elastoplastic behavior of two-phase particle-reinforced ductile matrix composites containing many randomly dispersed elastic spherical inhomogeneities. A complete second-order formulation is presented based on the probabilistic spatial distribution of spherical particles, explicit pairwise particle interactions (for both the elastic and plastic sub-problems), and the ensemble-volume averaging procedure. As a result, two alternative "effective yield functions" are derived micromechanically. The present work represents a significant improvement over the recent work of Ju and Chen (1994a) which is based on the first-order (noninteracting) stress perturbations on the matrix due to elastic particles. The derived ensemble-averaged yield criterion together with the assumed overall associative plastic flow rule and the hardening law then fully characterize the elastoplastic behavior of PRDMCs under any arbitrary three-dimensional loading/unloading histories. The present framework is at variance with most existing works in the micromechanics literature of PRDMCs, which are only applicable to monotonic, proportional loadings. In addition, the current work is completely different from all existing effective medium methods developed for PRDMCs since the former considers explicit inter-particle interactions and random particle distributions whereas the latter only considers one single particle embedded in an effective medium (and never actually considers particle locations, spatial distributions or explicit particle interactions).

The initial yield criteria for incompressible ductile matrix containing many identical spherical voids proposed by the present framework are compared with those proposed by Gurson (1977), Tvergaard (1981), Pont Castaneda (1991) and Qiu and Weng (1993). The proposed method is also applied to the special case of uniaxial stress loadings to predict the elastoplastic stress–strain responses. Moreover, the results are compared with the experimental data reported by Yang *et al.* (1991).

In a forthcoming paper, three-dimensional computational return mapping algorithms and finite element implementation of the proposed formulations will be presented. Specifically, three-dimensional strain-driven return mapping (backward Euler) algorithms, continuum and consistent elastoplastic tangent moduli, extension to elasto-viscoplastic formulation, and finite element examples will be systematically addressed.

Acknowledgements—This work was sponsored in part by the National Science Foundation, Mechanics and Materials Program, under PYI Grant MSS-9157238 (Program Monitor: Dr Oscar W. Dillon), by the CUREe-Kajima Phase II Research Project under Grant D950217 (KaTRI Monitors: Dr Y. Nojiri, Mr Y. Nobuta and Mr K. Horikoshi; CUREe Monitor: Professor Wilfred D. Iwan), and by the Air Base Systems Branch of the Flight Dynamics Directorate, Air Force Wright Laboratories (Program Monitor: Dr Jeff W. Rish, III). These supports are gratefully acknowledged.

REFERENCES

Bhattacharyya, A., Sakaki, T. and Weng, G. J. (1993). The influence of Martensite shape, concentration, and phase transformation strain on the deformation behavior of stable dual-phase steels. *Metallurgical Trans. A* **24A,** 301–314.

Dvorak, G. J. (1991). Plasticity theories for fibrous composite materials. In *Metal Matrix Composites: Mechanisms and Properties* (Eds R. K. Everett and R. J. Arsenault). Academic Press, Boston.

Eshelby, J. D. (1957). The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proc. R. Soc. Lond.* **A241,** 376–396.

Gurson, A. L. (1977). Continuum theory of ductile rupture by void nucleation and growth: part I—yield criteria and flow rules for porous ductile media. *Trans. A.S.M.E., J. Engng Mat. Tech.* **99,** 2–15.

Hill, R. (1963). Elastic properties of reinforced solids: some theoretical principles. *J. Mech. Phys. Solids* **11,** 357–372.

Hutchinson, J. W. (1970). Elastic–plastic behavior of polycrystalline metals and composites. *Proc. R. Soc. Lond.* **A319,** 247–272.

Hutchinson, J. W. (1976). Bounds and self-consistent estimates for creep of polycrystalline materials. *Proc. R. Soc. London.* **A348,** 101–127.

Ibrahim, I. A., Mohamed, F. A. and Lavernia, E. J. (1991). Particulate reinforced metal matrix composites—a review. *J. Mater. Sci.* **26,** 1137–1156.

Ju, J. W., Simo, J. C., Pister, K. S. and Taylor, R. L. (1987). A parameter estimation algorithm for inelastic material models. In *Proc. Second Int. Conf. on Constitutive Laws for Engineering Materials: Theory and Application*, pp. 1233–1240, 5–7 June 1987, University of Arizona.

Ju, J. W. and Chen, T. M. (1994a). Micromechanics and effective elastoplastic behavior of two-phase metal matrix composites. *J. Engng Mat. Tech.* **116,** 310–318.

Ju, J. W. and Chen, T. M. (1994b). Micromechanics and effective moduli of elastic composites containing randomly dispersed ellipsoidal inhomogeneities. *Acta Mechanica* **103,** 103–121.

Ju, J. W. and Chen, T. M. (1994c). Effective elastic moduli of two-phase composites containing randomly dispersed spherical inhomogeneities. *Acta Mechanica* **103**, 123–144.

Kroner, E. (1990). Modified Green function in the theory of heterogeneous and/or anisotropic linearly elastic media. In *Micromechanics and Inhomogeneity* (Eds G. J. Weng, M. Taya and H. Abe), pp. 197–211. Springer-Verlag, New York.

Lagoudas, D. C., Gavazzi, A. C. and Nigam, H. (1991). Elastoplastic behavior of metal matrix composites based on incremental plasticity and the Mori–Tanaka averaging scheme. *Computational Mech.* **8**, 193–203.

Leunberger, D. G. (1984). *Linear and Nonlinear Programming*, 2nd edn. Addison-Wesley, Massachusetts.

Levenberg, K. (1944). A method for the solution of certain problems in least squares. *Quart. Appl. Math.* **2**, 164–168.

Li, Y. Y. and Chen, Y. (1990). An incremental plastic analysis of multiphase materials. *J. Appl. Mech.* **57**, 562–568.

Marquardt, D. (1963). An algorithm for least-squares estimation of nonlinear parameters. *SIAM J. Appl. Math.* **11**, 431–441.

Mazilu, P. (1972). On the theory of linear elasticity in statistically homogeneous media. *Rev. Roum. Math. Pures et Appl.* **17**, 261–273.

Mura, T. (1987). *Micromechanics of Defects in Solids*, 2nd edn. Martinus Nijhoff, The Netherlands.

Ponte Castaneda, P. (1991). The effective mechanical properties of nonlinear isotropic composites. *J. Mech. Phys. Solids* **39**, 45–71.

Qiu, Y. P. and Weng, G. J. (1993) Plastic potential and yield function of porous materials with aligned and randomly oriented spherical voids. *Int. J. Plasticity* **9**, 271–290.

Simo, J. C., Ju, J. W., Pister, K. S. and Taylor, R. L. (1988). Assessment of cap model: consistent return algorithm and rate-dependent extension. *J. Engng Mech., ASCE* **114**, 191–218.

Suquet, P. M. (1993). Bounds and estimates for the overall properties of nonlinear composites. In *Proc. of the MECAMAT Conference Micromechanics of Materials* (Eds J. J. Marigo and G. Rousselier). Fontainbleau, France.

Tandon, G. P. and Weng, G. J. (1988). A theory of particle-reinforced plasticity. *J. Appl. Mech.* **55**, 126–135.

Tvergaard, V. (1981). Influence of voids on shear band instability under plane strain condition. *Int. J. Fracture* **17**, 389–407.

Tvergaard, V. (1982). On localization in ductile materials containing spherical voids. *Int. J. Fracture* **18**, 237–252.

Weng, G. J. (1990). The overall elastoplastic stress–strain relations of dual-phase metals. *J. Mech. Phys. Solids* **38**, 419–441.

Yang, J., Picard, S. M., Cady, C., Evans, A. G. and Mehrabian, R. (1991). The stress/strain behavior of aluminum matrix composites with discontinuous reinforcements. *Acta Metall. Mat.* **39**, 1863–1869.